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International Journal of Heat and Mass Transfer 49 (2006) 3746–3751

International Journal of HEAT and MASS TRANSFER

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Heat conduction in fractal tree-like branched networks

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Received 5 September 2005; received in revised form 14 December 2005
Available online 30 March 2006

Abstract

The geometric structures and fractal dimensions of fractal tree-like branched networks have the significant influence on the efficiency of physiological, communication and transport processes. We analyze the heat conduction through symmetric fractal tree-like branched networks. We obtain the expression of thermal conductivity in the networks and analyze the relationship between the effective thermal conductivity (ETC) and the geometric structures of the networks. We have found that the ETC of the networks is always less than that of a single channel, and the value of the thermal conductivity of the network can tend to zero under certain conditions; as long as the branching number N is fixed, the heat conduction reaches the fastest rate at the same diameter ratio β_m which is corresponding to the fractal dimension $D_d = 2.0$. We have also found that the heat conduction in the networks is rather different from Murray's law both for laminar regime ($2^{-1/3}$) and for turbulent flow regime ($2^{-3/7}$).

Keywords: Fractal tree-like branched networks; Effective thermal conductivity; Thermal–electrical analogy; Constructal

1. Introduction

The transports in dendritic structures of natural systems/networks such as mammalian circulatory and respiratory systems, leaves, river basins, energy networks, world-wide webs, internet and social networks are of considerable current interest [1–7]. It has been shown that the structures of these networks may have the fractal tree-like branched structures which can be space filling [8] and ensure minimal dissipation [9–13]. However, most of attentions are paid to mass transfer in the branched structures, and the heat conduction in the networks has not been studied thoroughly, especially, the relationship between the geometric structures and the properties of heat conduction is not well understood.

Tree-like branched networks play a unique role in physics, biology and engineering. The investigations of the tree-like branched network began from 1926 since Murray proposed Murray's law [14] for cardiovascular system. Numerous subsequent researches extended Murray's work. Noteworthy is West et al.'s [9] work, who discussed the origin of allometric scaling laws in biology in a rather general and detailed fashion. Bejan [13,15,16] developed "Constructal Theory" by optimizing the access between one point and a finite volume and applied the theory to the cooling of electronic devices and other engineering fields. The constructal principle is purely geometric, the time arrow of this construction is from small to large, and the most important is that the optimal result leads to the tree-like branched network. Based on the constructal theory, Bejan [17] predicted 3/4-power relation between body heat loss and body size from pure theory.

Fractals abound in nature. Fractal can not only describe the fractal objects, but also serve as the basis for designing equipment. Kearney [18] pointed out that equipment built with fractal characteristics could offer advantages over traditional fluid mixers and distributors. Chen and Cheng [11] studied the convective heat transfer and pressure drop in a fractal branched net of rectangular shape, and compared the network with the conventional parallel channels. They drew a conclusion that the fractal tree-like branched

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0017-9310/$ - see front matter
doi:10.1016/j.ijheatmasstransfer.2006.01.033

Nomenclature

A_e	equivalent cross-sectional area of the network
A_k	cross-sectional area of the kth level channel
d_k	branch diameter at the kth level
D_d	fractal dimension of channel diameter distribution
D_l	fractal dimension of channel length distribution
l_e	equivalent length of the network
l_k	branch length of the kth level
N	branching number
m	total number of branching levels
R	total thermal resistance of the entire network
R_k	thermal resistance of a single channel of the kth level
V	total volume of the network

Greek symbols

β	ratio of the diameter of the channel at the $(k+1)$th branch level to that at the kth branch level
β_m	diameter ratio at which the effective thermal conductivity (ETC) is greatest
γ	ratio of the length of the channel at the $(k+1)$th branch level to that at the kth branch level
λ	thermal conductivity of the channel material
λ_e	effective thermal conductivity (ETC)
λ^+	dimensionless effective thermal conductivity
Δ	diameter exponent

Subscripts

k	rank of channel
e	effective

network can increase the total convective heat transfer rate and reduce the total pressure drop in the fluid.

Although the fractal tree-like branched network has so many important physical properties, the relationship between the thermal properties and its geometrical features is not well understood. In this paper, we study the properties of heat conduction in the network using a more general model. We derive the analytical expressions of the thermal conductivity and discuss the relationship between the thermal conductivity and the geometrical parameters of the network. We also compare our results to the transport properties in botanical tree, human cardiovascular system and bronchial tree [14,19,20], which obey Murray's law.

2. The fundamental features of the fractal tree-like branched networks

Most distribution systems can be described by a branched network in which the sizes of tubes regularly decrease. The commonality of natural branched networks has also been recognized in the field of fractal geometry. In fractal geometry [8], many of the geometrical features of a natural branched network can be approximated by repeating a finite number that follows a postulated properly designed algorithm, which results in an increasing number of channels with smaller diameter and an increase in the total cross-sectional flow area. It has been shown that mass transfer in these tree-like networks is efficient; similarly, heat transfer in such network may have many advantages.

We can obtain the tree-like network structures of different branching levels as shown in Fig. 1(a) by repeating the elemental branch as shown in Fig. 1(b). We assume that each branch of the network is smooth cylinder, and thickness of the tube wall can be ignored. Suppose that every channel is divided into N branches at the next level (e.g., $N = 2$ in Fig. 1). The model presented here should be viewed as an idealized representation in which we ignore complications such as tapering of vessels and nonlinear effects. These play only a minor role in determining the properties of the entire network and could be incorporated in more detailed analysis of specific systems. A typical branch at some intermediate level k $(k = 0, 1, 2, \ldots)$ has

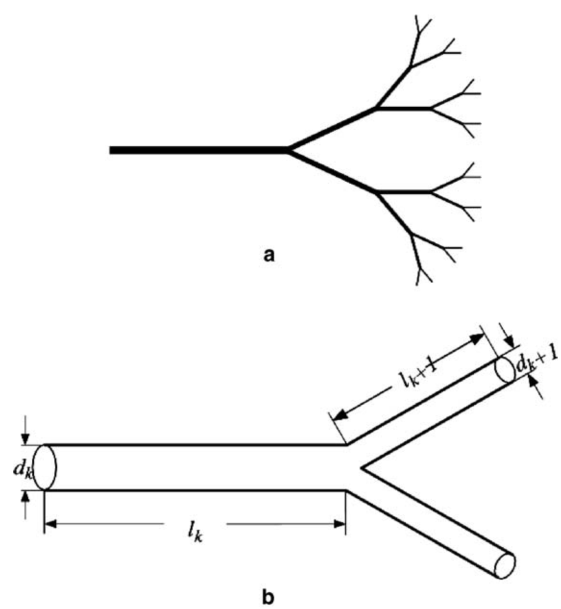

Fig. 1. Sketch of (a) a fractal tree-like branched network ($N = 2$, $m = 4$), and (b) the kth branching level with $N = 2$.

length l_k and diameter d_k (Fig. 1). To characterize the branch, we introduce scale factors as:

$$\gamma = l_{k+1}/l_k \quad \text{and} \quad \beta = d_{k+1}/d_k \ (k = 0, 1, 2, \ldots) \tag{1}$$

Thus it is easy to get

$$l_k = l_0\gamma^k \quad \text{and} \quad d_k = d_0\beta^k \tag{2}$$

where l_0 and d_0 are the length and diameter of the 0th branching level, respectively. According to the fractal characteristics of the structure [8], we have

$$N = \gamma^{-D_l} = \beta^{-D_d} \tag{3}$$

where D_l and D_d is the fractal dimensions of channel length and diameter distribution, respectively.

3. Heat conduction in the fractal tree-like branched networks

In this section, the effective thermal conductivity (ETC) by thermal–electrical analogy is derived. For convenience we assume that the network as shown in Fig. 1 is composed of the material of high thermal conductivity λ, which is much larger than that of the material around the channels. So, we can only consider the heat conduction along the channels and neglect the heat conduction in other directions. This means that the one dimensional heat flow model is applied in this work. The one dimensional heat flow model and the thermal–electrical analogy technique were applied to analyze the effective thermal conductivity of heterogeneous media such as porous media by many researches [22–25].

According to the Fourier's law, the thermal resistance of a single channel of the kth level channel can be expressed as $R_k = l_k/(\lambda A_k)$. Fig. 2 displays the thermal–electrical analogy network of thermal resistance, with $N = 2$, $m = 2$. The total thermal resistance of the entire network, is given by the Ohm's law model as

$$R = \sum_{k=0}^{m} \frac{R_k}{N^k} = \frac{4l_0}{\lambda\pi d_0^2} \frac{1-(\gamma/N\beta^2)^{m+1}}{1-\gamma/N\beta^2} \tag{4}$$

Inserting Eq. (3) into Eq. (4), we can get the relation between the total thermal resistance and the fractal dimensions as well as the network structures

$$R = \frac{4l_0}{\lambda\pi d_0^2} \frac{1-N^{(2/D_d-1/D_l-1)(m+1)}}{1-N^{(2/D_d-1/D_l-1)}} \tag{5}$$

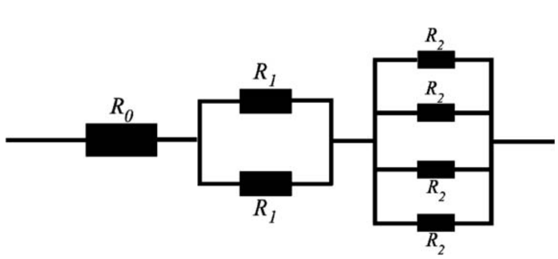

Fig. 2. The thermal–electrical analogy at $N = 2$ and $m = 2$.

The fractal tree-like branched network can be equivalent to a single channel, and then the thermal conductivity of the equivalent single channel is defined as the effective thermal conductivity (ETC) of the network. Then, the thermal resistance for the equivalent single channel can be written as

$$R = l_e/(\lambda_e A_e) \tag{6}$$

where l_e, A_e, λ_e are the equivalent length, cross-sectional area and thermal conductivity of the network, respectively. The length of the equivalent single channel is

$$l_e = \sum_{k=0}^{m} l_k = l_0 \frac{1-\gamma^{m+1}}{1-\gamma}.$$

The effective cross-sectional area is defined as the ratio of the total volume V to the effective length l_e of the network, i.e., $A_e = V/l_e$. Due to Eq. (2), the total volume is expressed as

$$V = \sum_{k=0}^{m} N^k V_k = \sum_{k=0}^{m} N^k \pi \left(\frac{d_k}{2}\right)^2 l_k = \frac{\pi d_0^2 l_0}{4} \frac{1-(N\beta^2\gamma)^{m+1}}{1-N\beta^2\gamma} \tag{7}$$

The effective cross-sectional area of the network is

$$A_e = V/l_e = \frac{\pi d_0^2}{4} \frac{1-\gamma}{1-\gamma^{m+1}} \frac{1-(N\beta^2\gamma)^{m+1}}{1-N\beta^2\gamma} \tag{8}$$

Then, the ETC of the network is

$$\lambda_e = \frac{l_e}{RA_e} = \lambda\left(\frac{1-\gamma^{m+1}}{1-\gamma}\right)^2 \frac{1-N\beta^2\gamma}{1-(N\beta^2\gamma)^{m+1}} \frac{1-\gamma/N\beta^2}{1-(\gamma/N\beta^2)^{m+1}} \tag{9}$$

Substituting Eq. (3) into Eq. (9), the ETC of the network can be also expressed as

$$\lambda_e = \lambda\left[\frac{1-N^{-(m+1)/D_l}}{1-N^{-1/D_l}}\right]^2 \frac{1-N^{(1-1/D_l-2/D_d)}}{1-N^{(1-1/D_l-2/D_d)(m+1)}} \times \frac{1-N^{(2/D_d-1/D_l-1)}}{1-N^{(2/D_d-1/D_l-1)(m+1)}} \tag{10}$$

From Eqs. (9) and (10), the dimensionless ETC is obtained as

$$\lambda^+ = \frac{\lambda_e}{\lambda} = \left(\frac{1-\gamma^{m+1}}{1-\gamma}\right)^2 \frac{1-N\beta^2\gamma}{1-(N\beta^2\gamma)^{m+1}} \frac{1-\gamma/N\beta^2}{1-(\gamma/N\beta^2)^{m+1}} \tag{11}$$

and

$$\lambda^+ = \frac{\lambda_e}{\lambda} = \left[\frac{1-N^{-(m+1)/D_l}}{1-N^{-1/D_l}}\right]^2 \frac{1-N^{(1-1/D_l-2/D_d)}}{1-N^{(1-1/D_l-2/D_d)(m+1)}} \times \frac{1-N^{(2/D_d-1/D_l-1)}}{1-N^{(2/D_d-1/D_l-1)(m+1)}} \tag{12}$$

Eqs. (11) and (12) reveal that the dimensionless ETC of the networks depends on the geometric structures.

4. Results and discussion

In this section, we compute the dimensionless ETC and determine the relationship among the ETC, the geometrical factors and fractal dimensions of the network. Fig. 3 compares the effect of length ratio γ, diameter ratio β, branching number N and total number of branching levels m on the ETC. It is found that small variations in the network geometrical structures can induce very large variations in the ETC, which is much similar to the net air flux in bronchial tree [20,21]. It is clear that the ETC is always smaller than that of the material of the channel. Fig. 3(a) and (b) show that the ETC of the network decreases as the increase of length ratio γ at the fixed N and m. Fig. 3(a) reveals, at $N = 2$ and $m = 3$, that there exists a same diameter ratio $\beta_m = 0.707$, at which the ETC is greatest, equal to the thermal conductivity of the channel material. Fig. 3(b) also reveals, at $N = 3$ and $m = 3$, that the ETC reaches the maximum value at $\beta_m = 0.577$, which is less than 0.707 as $N = 2$ and $m = 3$. Fig. 3(c) indicates that for the fixed branching number $N = 2$ and length ratio $\gamma = 0.6$, although the total number of branching levels m is different, there also exists the same diameter ratio $\beta_m = 0.707$, at which the ETC is greatest. It is also found from Fig. 3 that when $\beta < \beta_m$, the ETC increases as the increase of the diameter ratio β, but when $\beta > \beta_m$, the ETC decreases as the increase of the diameter ratio β.

Fig. 3 also reveals that the maximum dimensionless ETC is always 1.0, i.e., equal to that of channel material, and is independent of the branching number N, total number of branching levels m and length ratio γ. These results are consistent with the physical situations. The further results of Eq. (11) are shown in Figs. 4 and 5. Fig. 4 shows the relation between the ETC and branching number N, diameter ratio β at the total number of branching level $m = 3$ and length ratio $\gamma = 0.6$. Fig. 5 depicts the relation between the diameter ratio (β_m) of the maximum ETC versus the branching number N. From Figs. 3–5 it is also found that the diameter ratio (β_m) for the maximum ETC

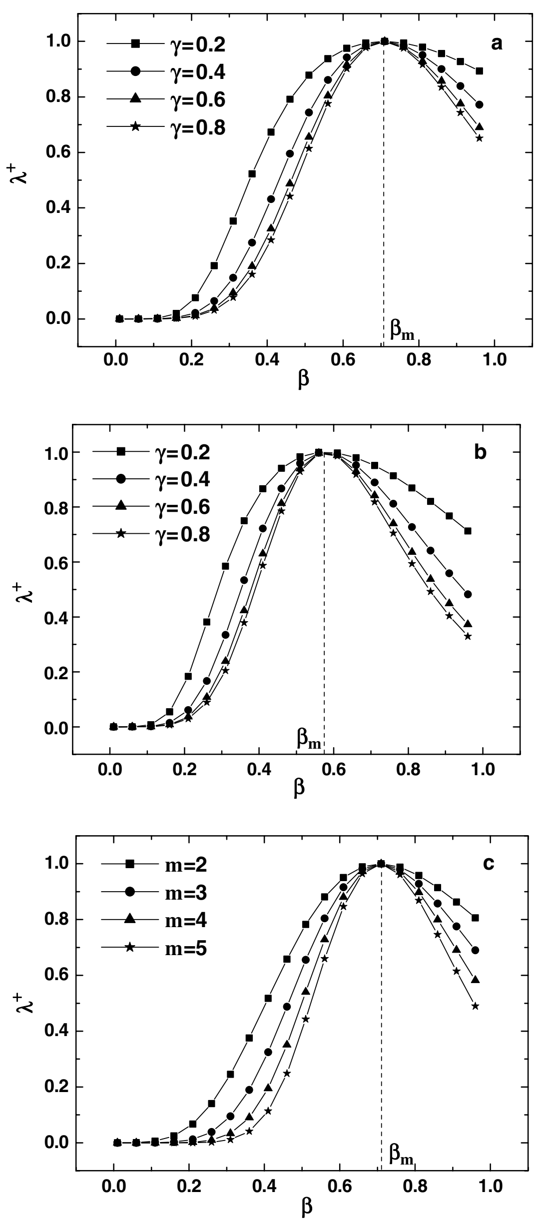

Fig. 3. The dimensionless effective thermal conductivity λ^+ versus (a) γ and β at $N = 2$ and $m = 3$, (b) γ and β at $N = 3$ and $m = 3$, and (c) β and m at $N = 2$ and $\gamma = 0.6$.

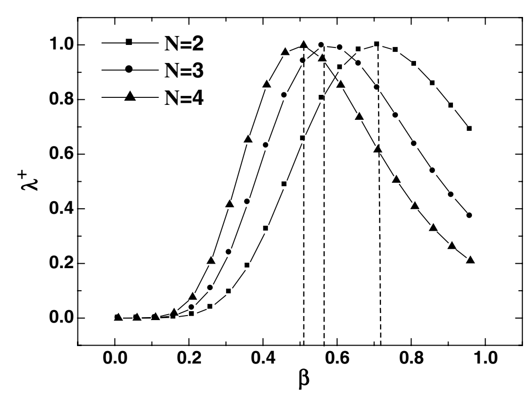

Fig. 4. The dimensionless effective thermal conductivity λ^+ versus β for different N at $m = 3$ and $\gamma = 0.6$.

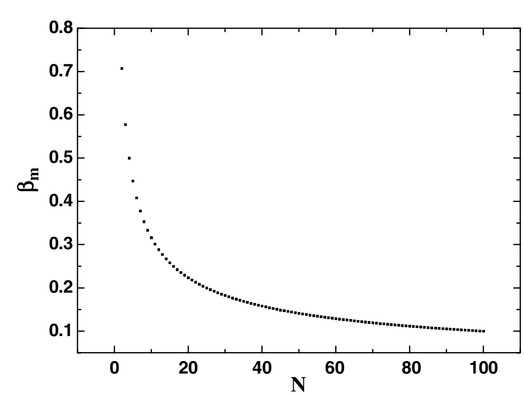

Fig. 5. β_m versus N.

is only related to the branching number N but is independent of the total number of branching levels m and length ratio γ, and the value of β_m decreases with the increase of branching number N. For $N = 2$ and $d_1 = d_2$ (two branching (daughter) channels have the same diameter in each level), the diameter ratio $\beta_m = 0.707 \approx 2^{-1/2}$ which exactly satisfy the relation [8] $d^{\Delta} = d_1^{\Delta} + d_2^{\Delta}$ (where d is the diameter of mother channel), where the diameter exponent being $\Delta = 2$. Furthermore, we find that the branching number N satisfies $\beta_m = N^{-1/\Delta}$ (where $\Delta = 2$, $N = 2, 3, 4, \ldots$) for the maximum thermal conductivity, see Fig. 5. For the water flow in botanical tree, blood flow in human cardiovascular system and airflow in bronchial tree [14,19,20], the optimized diameter ratio is close to the space-filling branching ratio $2^{-1/3} \cong 0.7937$ known as Murray's law [14]. It has been shown by Bejan et al. [26] that, in the laminar regime the tree conductivity is maximized with $\beta_m = 2^{-1/3}$ under the constraints of constant total duct volume and area allocated to the tree and the optimized diameter is $\beta_m = 2^{-3/7}$ for the turbulent flow regime. It is clear that the heat conduction in the networks is rather different from Murray's law both for laminar regime ($2^{-1/3}$) and for turbulent flow regime ($2^{-3/7}$).

Fig. 6 shows the dimensionless ETC versus the fractal dimensions of length and diameter distributions at different branching numbers N and total number of branching levels m. It is seen from Fig. 6 that for the fixed N and m, the larger the D_l (i.e., the larger the length ratio γ), the smaller the ETC , but the amplitude of variation is relatively small. For the different D_l, N and m, the ETC reaches the maximum value, which is equal to that of the channel material at the same fractal dimension $D_d = 2.0$. Further calculation indicates that $D_d = 2.0$ corresponds to the diameter ratio β_m (see Eq. (3)), i.e., the ETC reaches the maximum, which is equal to that of the channel material at the same fractal dimension $D_d = 2.0$, independent of N, m and D_l. The result $D_d = 2.0$ is consistent with the diameter exponent $\Delta = 2$ and independent of geometrical parameters, N, m and D_l.

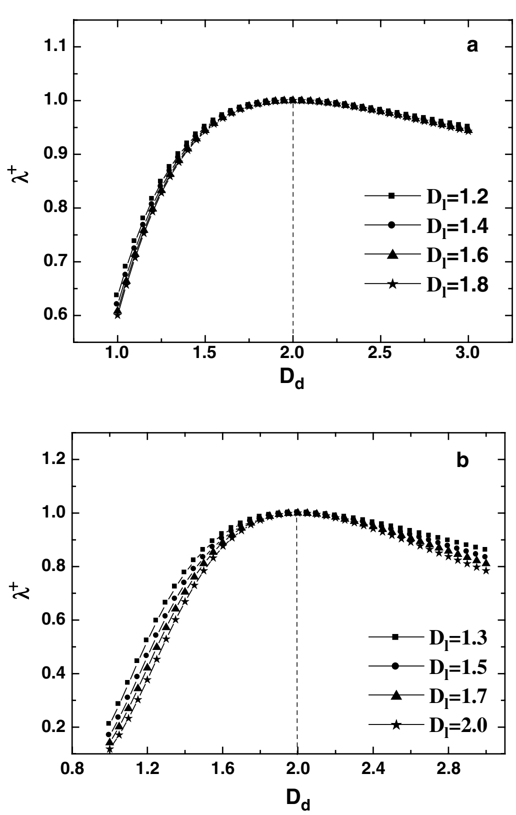

Fig. 6. The dimensionless effective thermal conductivity λ^+ versus (a) D_l and D_d at $N = 2$ and $m = 3$, and (b) D_l and D_d at $N = 3$ and $m = 5$.

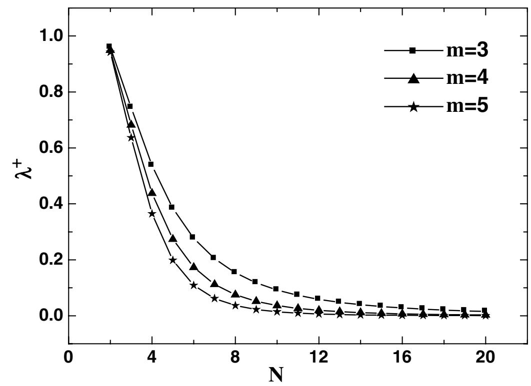

Fig. 7. The dimensionless effective thermal conductivity λ^+ versus N for different m at $\gamma = 0.4$ and $\beta = 0.8$.

The relation among the dimensionless ETC, branching number N and total number of branching levels m is shown in Fig. 7. The figure shows that for the fixed length ratio γ and diameter ratio β ($\gamma = 0.4$, $\beta = 0.8$), the dimensionless ETC decreases with the increase of N and m. It is interesting to note that the ETC of the networks may tend to zero when increasing the branching number N. This implies that the thermal resistance of heat conductivity through the fractal tree-like branched structures increases rapidly with the branching number N.

5. Concluding remarks

We have derived the expression of the ETC and determined the relation among the ETC, the geometric structures and the fractal dimensions of the fractal tree-like branched model. We have found that (1) the ETC of the networks is always less than that of a single channel, and the value of the thermal conductivity of the network can tend to zero under certain conditions; (2) the longer the branches (i.e., the larger the fractal dimension of length distribution D_l), the smaller the ETC; (3) as long as the branching number N is fixed, the heat conduction reaches the fastest rate (meaning the highest ETC) at the same diameter ratio β_m in spite of different total number of branching levels m and length ratio γ, and the maximum of thermal conductivity equals that of a single channel. Moreover, the position of the β_m decreases with the increase of the branching number N; (4) the ETC is the greatest at fractal dimension $D_d = 2.0$, which is corresponding to the diameter ratio β_m, independent of other geometrical parameters. The maximum is equal to the thermal conductivity of the channel material; (5) the more the branches of the network, the higher the thermal resistance of the networks. We have also found that the heat conduction in the networks is rather different from Murray's both for laminar regime ($2^{-1/3}$) and for turbulent flow regime ($2^{-3/7}$), and the maximum thermal conductivity satisfies $\beta_m = N^{-1/\Delta}$ for the branching number N (where $\Delta = 2$, $N = 2, 3, 4, \ldots$) for the networks.

We have shown that heat conduction is similar to mass transfer in fractal tree-like branched structures. Since the self-avoidance narrows the choice of branching angle and the total number of branching levels, the values of which are thus restricted [8]. The results of this work show that the properties of heat transfer of the networks may be widely applied to the cooling of electronic devices, bioengineering and biotechnology, materials of space equipments etc. Under certain conditions, the property of low thermal conductivity is also very important for designing insulation materials and structures such as space equipments.

Acknowledgement

This work was supported by the National Natural Science Foundation of China through grant number 10572052.

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